BULK VALUE OF THE CENTRAL CHARGE IN CONFORMALLY INVARIANT CRITICAL PHENOMENA

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The concept of the central charge in a two-dimensional conformally invariant critical system is discussed. It is shown how this concept is related to a hyperuniversal amplitude ratio. In turn we illustrate the computation of its value by means of high-temperature series expansions for the cases of the Ising model and the three-state Potts model. The results are quite satisfactory and agree with the known results in these sample cases.

The idea of conformal invariance in critical phenomena stems from scaling ideas [1]. Suppose we have two blocks of spins of size D, separated by a distance R from each other, as in Fig. 1. Near the critical point, where the spin-spin correlations are strong, we expect that the cell-cell correlations will act a lot like spin-spin correlations. For the Gibbs potential for example, we might expect that, in d dimensions,

$$\tilde{G}(\widetilde{K_c - K}, \tilde{H}) = D^d \tilde{G}(K_c - K, H), \tag{1}$$

where naively,

$$\widetilde{K_c - K} = (K_c - K)D^d, \quad \tilde{H} = D^d H. \tag{2}$$

Here we denote the magnetic field by H and $K = J/kT$, where J is the exchange integral, k is Boltzmann's constant and T is the temperature. But the correlations aren't perfect, so we introduce the parameters x and y such that,

$$\utilde{K_c - K} = D^y(K_c - K), \quad \tilde{H} = D^x H, \tag{3}$$

such that

$$G(D^y(K_c - K), D^x H) = D^d G(K_c - K, H). \tag{4}$$

If we pick the cell size, $D = (K_c - K)^{-1/y}$, then (4) becomes,

$$G(K_c - K, H) = (K_c - K)^{d/y} G\left(1, \frac{H}{(K_c - K)^{x/y}}\right). \tag{5}$$

From standard thermodynamics and (5) we obtain the relations

$$2 - \alpha = d/y, \quad x/y = \Delta = \gamma + \beta, \tag{6}$$

where α is the index of divergence of the specific heat, γ is the index of divergence of the magnetic susceptibility, β is the index of convergence of the spontaneous magnetization, and Δ is the "gap" critical index which refers to the rate of closure of the gap in the distribution of zeros of the partition function in the complex activity plane as the temperature approaches the critical temperature, $K \to K_c$. Of course, we can rewrite the scaling parameters, x and y in terms of the critical indices as,

$$y = d/(2 - \alpha), \quad x = d\Delta/(2 - \alpha). \tag{7}$$

The spin-spin correlation function can be written as,

$$g(r, K) = \tfrac{1}{4}\langle(\sigma_{\vec{r}} - \langle\sigma_{\vec{r}}\rangle)(\sigma_{\vec{0}} - \langle\sigma_{\vec{0}}\rangle)\rangle, \tag{8}$$

where the spins are normalized so the $\langle\sigma^2\rangle = 1$. For cells, we want roughly to keep this sort of normalization. Thus we write,

$$D^z \tilde{s} = D^{-d} \sum_{\text{cell}} \sigma_j, \tag{9}$$

and

$$\tilde{g}(\tilde{r}, \widetilde{K_c - K}) = \frac{1}{4}\left\langle (\tilde{s}_{\vec{\tilde{r}}} - \langle \tilde{s}_{\vec{\tilde{r}}}\rangle)(\tilde{s}_{\vec{\tilde{0}}} - \langle \tilde{s}_{\vec{\tilde{0}}}\rangle)\right\rangle. \tag{10}$$

From the magnetic index, $Hd^x = \tilde{H} = HD^{d+z}$, we can conclude the relation $z = d - x$ between our parameters, and so we can reduce the scaling version of the spin-spin correlation function to

$$g(r, K_c - K) = (K_c - K)^{2(d-x)/y} g(r(K_c - K)^{1/y}, 1). \tag{11}$$

by comparison with the definition of the critical index η for the asymptotic behavior of the spin-spin correlation function for large r, we can also get the relation,

$$x = \frac{d + 2 - \eta}{2}. \tag{12}$$

After this brief review of the rudiments of scaling theory, our next step [2,3] is to go beyond "scale invariance" in the scaling ideas to "local scale invariance" where the change of scale is not uniform, however angles are still preserved. That is to say, "conformal invariance." Just as scaling invariance was a hypothesis whose consequences we looked at above, so too "conformal invariance" of (for example) the spin-spin correlation function is a hypothesis. Here we are speaking strictly at $T = T_c$ because we know that the simple conformal transformation of rotational invariance fails for the 2-dimensional Ising model for $T \neq T_c$.

For $d > 2$, the only conformal transformations are: translation, rotation, dilation and inversion, *i.e.*, $x_i' = x_i/x^2$. These operations form a rather small conformal group, but still some conclusions can be drawn from the assumptions of conformal invariance. For example, if we map, $\vec{r'} = l\vec{r}$, then

$$\langle \phi_a(\vec{r}_1)\phi_b(\vec{r}_2)\cdots\rangle_C = l^{\omega^* - x_a - x_b \cdots}\langle \phi_a(\vec{r_1'})\phi_b(\vec{r_2'})\cdots\rangle_C, \tag{13}$$

where the subscript C denotes the connected part and ω^* denotes Fisher's [4] anomalous dimension of the vacuum.

For the case of $d = 2$ this theory is related to an extremely large conformal group, that is to say, any analytic function generates a conformal mapping of the complex plane. We need to study the algebraic structure to get some information out of it however. It will involve a special case of the Lie algebras called the Virasoro [5] algebra. The two dimensional plane can be described by the complex variable $z = x + iy$. It is convenient to introduce,

$$z = \xi_1 + i\xi_2, \quad \bar{z} = \xi_1 - i\xi_2, \tag{14}$$

with the metric,

$$ds^2 = dz\, d\bar{z}. \tag{15}$$

The conformal group in 2-dimensions consists of

$$z \mapsto \zeta(z), \quad \bar{z} \mapsto \bar{\zeta}(\bar{z}), \tag{16}$$

where ζ and $\bar{\zeta}$ are arbitrary functions, and is the direct product,

$$\mathcal{G} = \Gamma \otimes \bar{\Gamma}. \tag{17}$$

The "infinitesimal" transformations of Γ are,

$$z \mapsto z + \epsilon(z), \tag{18}$$

where

$$\epsilon(z) = \sum_{-\infty}^{+\infty} \epsilon_n z^{n+1}, \tag{19}$$

with the $\epsilon_n << 1$. From "local scale invariance" we expect, (for ϕ's of simple structure)

$$\begin{aligned}\langle X \rangle \equiv \langle \phi_a(z_1, \bar{z}_1)\phi_b(z_2, \bar{z}_2)\cdots\rangle_C \\ = \left[\left(\frac{d\zeta(z_1)}{dz_1}\right)^{\Delta_a}\left(\frac{d\bar{\zeta}(\bar{z}_1)}{d\bar{z}_1}\right)^{\bar{\Delta}_a}\right]\left[\left(\frac{d\zeta(z_2)}{dz_2}\right)^{\Delta_b}\left(\frac{d\bar{\zeta}(\bar{z}_2)}{d\bar{z}_2}\right)^{\bar{\Delta}_b}\right]\cdots \\ \times \langle \phi_a(z_1 + \epsilon(z_1), \bar{z}_1 + \bar{\epsilon}(\bar{z}_1))\phi_b(z_2 + \epsilon(z_2), \bar{z}_2 + \bar{\epsilon}(\bar{z}_2))\cdots\rangle_C\,, \end{aligned} \tag{20}$$

where here Δ_a replaces x_a in (13) and we take $\omega^* = 0$. Thus to first order in ϵ ($\bar{\epsilon}$ works the same way)

$$\phi_a(z_1,\bar{z}_1) \mapsto \phi_a(z_1,\bar{z}_1) + \epsilon(z_1)\frac{\partial}{\partial z_1}\phi_a(z_1,\bar{z}_1) + \Delta_a \epsilon'(z_1)\phi_a(z_1,\bar{z}_1). \tag{21}$$

Let us introduce the operators $T(z)$ [and of course $\bar{T}(\bar{z})$], such that to first order in $\epsilon(z)$,

$$\begin{aligned} \delta_\epsilon\langle X\rangle &= \delta_\epsilon\langle \phi_a(z_1,\bar{z}_1)\phi_b(z_2,\bar{z}_2)\cdots\rangle_C \\ &= \sum_i \left[\epsilon(z_i)\frac{\partial}{\partial z_i} + \epsilon'(z_i)\Delta_{a_i}\right]\langle \phi_a(z_1,\bar{z}_1)\phi_b(z_2,\bar{z}_2)\cdots\rangle_C \\ &= \frac{1}{2\pi i}\oint d\zeta\, \epsilon(\zeta)\langle T(\zeta)\phi_a(z_1,\bar{z}_1)\phi_b(z_2,\bar{z}_2)\cdots\rangle_C, \end{aligned} \tag{22}$$

where the contour encloses the points $z_1, z_2, \ldots$. By Cauchy's theorem,

$$\langle T(\zeta)\phi_a(z_1,\bar{z}_1)\phi_b(z_2,\bar{z}_2)\cdots\rangle_C = \sum_i \left[\frac{\Delta_a}{(\zeta - z_i)^2} + \frac{1}{\zeta - z_i}\frac{\partial}{\partial z_i}\right]\langle \phi_a(z_1,\bar{z}_1)\phi_b(z_2,\bar{z}_2)\cdots\rangle_C. \tag{23}$$

If we expand,

$$T(z) \equiv \sum_{n=-\infty}^{+\infty} z^{-(2+n)}L_n, \text{ then for } T_\epsilon \equiv \sum_{n=-\infty}^{+\infty}\epsilon_n L_n = \frac{1}{2\pi i}\oint \epsilon(\zeta)T(\zeta)\,d\zeta, \tag{24}$$

we have,

$$\delta_\epsilon\langle X\rangle_C = \langle T_\epsilon X\rangle_C. \tag{25}$$

Now we can compute the commutator of T and T_ϵ. We note from (23) that the most singular term will be proportional to $(\zeta - z_i)^{-4}$. Since the transformations we are dealing with are conformal, and therefore preserve angles, the terms in $\epsilon^{(n)}(z)$ are all determined through $n = 2$. Thus we can compute, after noting that $\Delta_T = 2$ by analysis of the same sort as explained above that,

$$[T_\epsilon, T(z)] = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \tfrac{1}{12}c\epsilon'''(z). \tag{26}$$

The last term is not completely specified by conformal invariance alone but depends on which class, characterized by the new parameter c, the transformation belongs to. Another way to note the role of c is to observe that the operators T and $\bar{T}$, which are the generators of infinitesimal conformal transformations, are linear combinations of the components of the energy-momentum tensor $T_{\mu,\nu}$. For a conformally invariant theory this tensor is traceless, *i.e.*

$$\Theta(z,\bar{z}) \equiv T_{xx} + T_{yy} = 0. \tag{27}$$

Furthermore, T only depends on z and $\bar{T}$ on $\bar{z}$. It has dimension 2 which means its correlations decay asymptotically with distance as

$$\langle T(z)T(0)\rangle = \frac{c}{2z^4}. \tag{28}$$

Thus the parameter c is the amplitude for this decay.

Eq. (26) is equivalent to the relation between the generators of the algebra, L_n,

$$[L_n, L_m] = (m-n)L_{m+n} + \tfrac{1}{12}c(m^3-m)\delta_{m,-m}\mathcal{I}, \tag{29}$$

where $\mathcal{I}$ is the unit element. In this form we recognize the generators as being the generators of a Virasoro algebra which has the central charge, c. The element $\mathcal{I}$ alone is the center (all the elements which commute with every element are called the center of an algebra) of this algebra. The algebra is called the central extension of the case where $c = 0$. By the analysis of the properties of this algebra, Friedan, *et al.* [6] found through the use of the Kac [7] formula and positivity that the only allowed values of c are,

$$c \geq 1, \qquad \text{or} \qquad c = 1 - \frac{6}{m(m+1)}, \quad m = 2,3,4,\ldots. \tag{30}$$

The allowed values of the Δ's are correspondingly given by

$$\Delta_{p,q} = \frac{[(m+1)p - mq)]^2 - 1}{4m(m+1)}, \quad p = 1,2,\ldots,m-1, \quad q = 1,2,\ldots,p. \tag{31}$$

Until recently, there has been no way to compute directly (except for finite size effects) the value of c. It has been deduced in several models by sorting through the possible values of the critical exponents to see which possible c matched the computed exponents for a given model. Since c is a fundamental bulk property, these approachs are unsatisfactory from a theoretical point of view. Here we shall use the work of Zamolodchikov [8] and Cardy [9] to calculate directly the bulk value of the central charge using high temperature expansions. The key lies in extending the notion of central charge away from the conformally invariant fixed point (FP), into the scaling regime. As one moves away from the conformally invariant point, T (and similarly $\bar{T}$) becomes a function of z and $\bar{z}$ and the trace of the energy-momentum tensor $\Theta(z,\bar{z})$ becomes non-zero. Thus Eq. (28) is replaced by a set of three equations

$$\langle T(z,\bar{z})T(0,0)\rangle = F(R)/z^4 \tag{32}$$

$$\langle T(z,\bar{z})\Theta(0,0)\rangle = \langle \Theta(z,\bar{z})T(0,0)\rangle = G(R)/z^3\bar{z} \tag{33}$$

$$\langle \Theta(z,\bar{z})\Theta(0,0)\rangle_C = H(R)/z^2\bar{z}^2 \tag{34}$$

Here F, G, and H are scaling functions which depend on the distance $R = (z\bar{z})^{1/2}$ and reduce to $c/2$, 0 and 0, respectively at the conformally invariant point. Since, the energy momentum tensor is conserved, these scaling functions are related to each other by the relation [8,9],

$$R\frac{dC}{dR} = -\tfrac{3}{2}H, \tag{35}$$

where $C = 2F - G - \frac{3}{8}H$. From (34) H is seen to be non-negative and hence C is a non-increasing function of R. It also equals c at the FP. This is Zamolodchikov's celebrated function, which is always decreasing along RG trajectories and shows that the RG flows always take one from a higher value of the central charge to a lower one.

For a Hamiltonian near a critical point of the form $\mathcal{H} = \mathcal{H}^* + t\int \epsilon(r) d^2r$, where $\mathcal{H}^*$ is the fixed point Hamiltonian, the trace Θ is given by [9],

$$\Theta(r) = 2\pi t(2 - x_\epsilon)\epsilon(r), \tag{36}$$

where x_ϵ is the scaling dimension of the energy ϵ. Thus integrating (35) along a trajectory which flows from the neighborhood of the critical FP with $C = c$ to the trivial FP with $C = 0$, we obtain the relation

$$c = 6\pi^2 t^2 (2 - x_\epsilon)^2 \int_0^\infty R^3 \langle \epsilon(R)\epsilon(0)\rangle_c dR \tag{37}$$

Following Singh and Baker [10], we now consider a lattice statistical model with energies e_i specified on the bonds of the lattice, and define the moments of the energy-energy correlation function as

$$\mu_{E,m} = \frac{1}{N}\sum_i \sum_j r_{i,j}^m (\langle e_i e_j \rangle - \langle e_i \rangle \langle e_j \rangle). \tag{38}$$

Using Eq (37) and the relation $2 - x_\epsilon = 2/(2 - \alpha)$, we can now express c as a hyper-universal amplitude ratio

$$c = \lim_{K \to K_c} \frac{12\pi (K_c - K)^2}{(2 - \alpha)^2} \mu_{E,2}. \tag{39}$$

For the case $\alpha > 0$ this result leads to the physically appealing interpretation that, aside from a constant, c is the singular part of the free-energy per correlation volume [9]. That is, rewriting (39),

$$c = \lim_{K \to K_c} -12\pi \frac{(1 - \alpha)}{(2 - \alpha)} f_s \xi_E^2, \tag{40}$$

Here f_s is in units of kT_c and ξ_E is in units of lattice spacing.

This quantity can be evaluated by developing high temperature series expansions for $\mu_{E,2}$. These series have been computed by Singh and Baker [10] for the Ising model (2-state Potts model) and for the 3-state Potts model in terms of the convenient expansion variable,

$$v = \frac{e^K - 1}{e^K - 1 + q}, \tag{41}$$

where q is the number of states in the Potts model. They obtained,

$$\mu_{E,2} = 2v^2 + 20v^4 + 162v^6 + 1200v^8 + 8462v^{10} + 57804v^{12}$$

$$+386102v^{14} + \cdots, \quad (q = 2) \tag{42a}$$

$$\mu_{E,2} = \frac{16}{9}v^2 + \frac{32}{9}v^3 + 16v^4 + \frac{544}{9}v^5 + \frac{1312}{9}v^6 + \frac{5056}{9}v^7 + \frac{15488}{9}v^8 + 4448v^9 +$$

$$\frac{147824}{9}v^{10} + \frac{384064}{9}v^{11} + \frac{1172696}{9}v^{12} + \frac{3651824}{9}v^{13} + \cdots, \ (q = 3) \tag{42b}$$

The critical points for the q-state Potts models are given by,

$$e^{K_c} - 1 = \sqrt{q}, \quad v_c = \frac{\sqrt{q}}{q + \sqrt{q}}. \tag{43}$$

By an appropriate change of variables, a good form for analysis for the present cases where the critical temperatures are exactly known, is,

$$c = \frac{12\pi}{(2-\alpha)^2}\left[\lim_{v \to v_c} \mu_{E,2}\left(1 - \frac{v}{v_c}\right)^2\right], \quad (q = 3), \tag{44a}$$

or

$$c = \frac{3\pi}{(2-\alpha)^2}\left[\lim_{v \to v_c} \mu_{E,2}\left(1 - \frac{v^2}{v_c^2}\right)^2\right], \quad (q = 2), \tag{44b}$$

as the series for $q = 3$ is in v and that for $q = 2$ is in v^2. Direct analysis of these series by the integral approximant method [3] leads to rapidly converging results for the Ising model and reasonably well converging results for the 3-state Potts model. We obtain the estimates,

$$c = 0.5 \pm 0.001 \quad (q = 2), \text{ and } \quad c = 0.80 \pm 0.01 \quad (q = 3). \tag{45}$$

These results correspond to the cases $m = 3$ and $m = 5$ of (30) respectively, which is most satisfactory and agrees with the known results.

In cases where we do not have the exact value of the critical temperature, but only series estimates, we have found it expeditious to replace

$$(K_c - K)^2 \to \left(\frac{\mu}{\frac{\partial \mu}{\partial K}} \right)^2 , \tag{46}$$

and so we are lead to look at the function,

$$c^*(K) = \frac{48\pi \mu_{E,2}}{(2-\alpha)^2} \left(\frac{\mu_{E,2}}{\frac{\partial \mu_{E,2}}{\partial K}} \right)^2 , \tag{47}$$

with the result,

$$c = \lim_{K \to K_c} c^*(K). \tag{48}$$

Summing this series with ordinary Padé approximants, rather than the more sophisticated integral approximants, we obtained for the case $q = 3$, the result $c = 0.80 \pm 0.01$ as before.

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Figure Captions

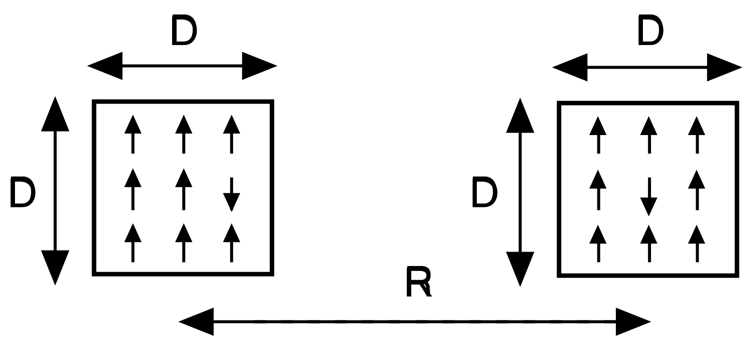

Fig. 1 Two strongly correlated $D \times D$ blocks at a distance R.